Statistical analysis of pulse propagation driven by polarization-mode dispersion

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The linear propagation of pulses driven by random polarization-mode dispersion is considered. Analytical expressions are derived for the probability-density functions of the pulse width, timing displacement, and degree of polarization. The study is performed in Stokes space, and frequency correlation between modes is shown to play an important role in it. © 2002 Optical Society of America

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1. INTRODUCTION

The study of pulse propagation in a fiber with random birefringence has become of great interest for telecommunication applications. Recent experiments have shown that polarization-mode dispersion (PMD) is one of the main limitations on fiber transmission links.[1] PMD occurs because single-mode fibers are not truly single mode but can exhibit different group velocities. PMD is thus characterized by a differential group delay (DGD) between the two modes associated with the fiber. PMD results in the splitting of a polarized input pulse into two orthogonal polarizations that arrive at the output at different times.

The effects of PMD are usually treated by means of the three-dimensional PMD vector. PMD vector $\hat{\mathbf{r}}$ gives the relation in Stokes space between the output state and the frequency derivative of the output state: $\hat{\mathbf{s}}'(\omega) = \hat{\mathbf{r}}(\omega) \times \hat{\mathbf{s}}(\omega)$. The principal states of polarization are defined as the states that satisfy $\hat{\mathbf{r}}(\omega) \times \hat{\mathbf{s}}(\omega) = \mathbf{0}$, so no change in output polarization can be observed close to these states at first order in ω. However, for broadband pulses the first-order approach is not valid anymore; the changes in the principal states of polarization and in the length of $\hat{\mathbf{r}}(\omega)$, i.e., the DGD, with respect to frequency cannot be neglected. As a consequence the frequency dependences of the PMD vectors and the DGD are worth studying.

For a given frequency the properties of the PMD vector are well known.[2–4] In particular, the elements of the vector are independent Gaussian random variables. Despite their obvious relevance, the frequency dependences of the PMD vectors and the DGD have been treated only in a few papers.[5,6] Furthermore, each paper deals with the computation of one particular expectation: the expectation of correlation between two PMD vectors[5] or the expectation of correlation between two square DGDs.[6] We are able to extend these results by using an approach based on the theory of stochastic differential equations. This theory gives us the means to compute any expectation of any combination of PMD vectors or DGDs at different frequencies. Furthermore, by applying and extending the theory of moments[7] we deduce from these formulas a closed-form expression for the means, variances, and probability-density functions of relevant quantities (such as time displacement, time rms width, and degree of polarization) for the characterization of pulse propagation.

2. DESCRIPTION OF THE MODEL

The evolution of polarized fields in randomly birefringent fibers is governed by the coupled Schrödinger equations with random PMD between two modes (polarizations)[8]:

$$i\mathbf{A}_z + K_0\mathbf{A} + iK_1\mathbf{A}_t - \frac{\beta''}{2}\mathbf{A}_{tt} = 0, \tag{1}$$

where subscripts stand for partial differentiation with respect to corresponding variables and $\mathbf{A}$ is the column vector that denotes the envelopes of the electric field in the two eigenmodes. The matrices K_0 and K_1 describe random fiber birefringence. The group-velocity dispersion (GVD) coefficient β'' is the second derivative of the propagation constant with respect to frequency. We can eliminate the fast random birefringence variations that appear in Eq. (1) by means of a change of variables, which leads to the new vector equation

$$i\mathbf{U}_z - \frac{\beta''}{2}\mathbf{U}_{tt} = iR\mathbf{U}_t, \tag{2}$$

where $\mathbf{U} \equiv M^{-1}\mathbf{A}$. $\mathbf{U} = (u, v)^T$ represents the slow evolution of the field envelopes in the reference frame of the local polarization eigenmodes, and matrix M obeys the equation $iM_z + K_0M = 0$. R is a z-dependent matrix that involves high-order PMD. It is associated with the coupling between the modes as well as with an accumulation of a mismatch between their phases.

The most commonly used model is the so-called retarded-plate model.[9,10] Birefringence strength $\Delta\beta$ and its derivative $\Delta\beta'$ are constant; the birefringence angle is constant over elementary intervals with length Δz; at junctions between the fiber pieces with length Δz, a random axial rotation is added as well as a random phase difference between the two field components, so the Stokes vector obeys a random walk over the Poincaré sphere. For our numerical simulations we use this model. If Δz is small compared to the dispersion length we can model this configuration by considering that the matrix R is[10]

$$R(z) = m_1(z)\Sigma_1 + m_2(z)\Sigma_2 + m_3(z)\Sigma_3, \tag{3}$$

where Σ_j are Pauli matrices:

$$\Sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \Sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \Sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and the real-valued processes m_j are random white noises with autocorrelation functions

$$\langle m_j(z) m_j(z') \rangle = \sigma^2 \delta(z - z'), \qquad \sigma^2 = \frac{\Delta\beta'^2 \Delta z}{12}. \tag{4}$$

In this model the so-called PMD parameter is given by $D_p = \sqrt{8/(3\pi)}\Delta\beta' \sqrt{\Delta z}$.[10] Our parameter σ^2 is simply related to D_p through $\sigma^2 = (\pi/32) D_p{}^2$. All models of random birefringence are eventually described by the white-noise model [Eq. (3)] with some effective parameter σ (or D_p) as soon as the correlation length of the random fluctuations of birefringence parameters is much smaller than the other characteristic lengths of the problem.[4,11,12]

For consistency, note that the usual GVD parameter is $D = -2\pi c \beta''/\lambda^2$, where λ is the carrier wavelength of the pulse (1.55 μm for standard optical fiber applications). Typical values of PMD parameter D_p have been measured in the range $0.1-1$ ps/$\sqrt{\text{km}}$.[13,14] Dispersion-shifted fibers (which are of particular interest for telecommunication) have been found to have particularly high values.[14] The correlation length Δz of PMD varies from 0.1 to 1 km, and the absolute value of GVD parameter D is 1–20 (ps/nm)/km.

3. PULSE CHARACTERIZATION

Numerical simulations were carried out for determination of the width and the degree of polarization of the pulse relative to the propagation distance.[9,15] In these simulations the effect of GVD was usually neglected to permit the effect of PMD to be examined exclusively. Empirical formulas for the pulse width and the degree of polarization were proposed. Furthermore, the number of samples necessary for stable averaged values was found to be large, which tends to prove that the variances of the pulse width and the degree of polarization are large as well. Below, we confirm these observations theoretically. In this section we properly define the time displacement, pulse width, and degree of polarization of a pulse. We include time displacement because fiber PMD induces not only optical pulse broadening but also differences in pulse arrival times for different fiber configurations.

First we introduce the time displacement, T_c:

$$T_c = \frac{\int t(|u|^2 + |v|^2)\mathrm{d}t}{\int (|u|^2 + |v|^2)\mathrm{d}t}. \tag{5}$$

If the initial pulse is polarized along the u axis, then the rotation of the pulse polarization can be characterized by the parameter P_r, defined as the ratio of the energy on the u axis to the total energy:

$$P_r = \frac{\int |u|^2 \mathrm{d}t}{\int (|u|^2 + |v|^2)\mathrm{d}t}. \tag{6}$$

The degree of polarization is defined by[16]

$$P_d = (s_1{}^2 + s_2{}^2 + s_3{}^3)^{1/2} \tag{7}$$

in terms of the following Stokes parameters:

$$E_0 = \int (|u|^2 + |v|^2(t)\mathrm{d}t,$$

$$s_1 = \int (|u|^2 - |v|^2)(t)\mathrm{d}t/E_0,$$

$$s_2 = 2\int \mathrm{Re}(u^* v)(t)\mathrm{d}t/E_0,$$

$$s_3 = 2\int \mathrm{Im}(u^* v)(t)\mathrm{d}t/E_0. \tag{8}$$

We can also introduce the degree of polarization in the following way: We construct the new field components ($u_{\theta,\xi}$ and $v_{\theta,\xi}$) as

$$u_{\theta,\xi}(t) = \cos(\theta)u(t) + \sin(\theta)e^{i\xi}v(t),$$

$$v_{\theta,\xi}(t) = -\sin(\theta)u(t) + \cos(\theta)e^{i\xi}v(t).$$

The energy in the component $u_{\theta,\xi}$ is $(1/2)E_0 + (1/2) \times \cos(2\theta)E_0 s_1 + (1/2)\sin(2\theta)[\cos(\xi)E_0 s_2 - \sin(\xi)E_0 s_3]$. We choose θ and ξ such as to maximize the energy in the component $u_{\theta,\xi}$: $\xi = -\arctan(|s_3|/|s_2|)$ and $2\theta = \arctan(\sqrt{s_2{}^2 + s_3{}^2}/|s_1|)$. The corresponding angle θ is the so-called polarization angle; the ratio of the energy of component $u_{\theta,\xi}$ to the total energy,

$$P_{d1} = \frac{\int |u_{\theta,\xi}|^2 \mathrm{d}t}{\int (|u|^2 + |v|^2)\mathrm{d}t}, \tag{9}$$

is then related to the degree of polarization through the simple identity $P_d = 2P_{d1} - 1$. Note that $P_{d1} \geqslant P_r$. Furthermore, the degree of polarization of a monochromatic pulse is always equal to 1, whatever the rotation of polarization. However, the spectral components of a short pulse tend to lose their polarization coherence, so the degree of polarization P_d of the corresponding pulse decays to zero.

It is not clear whether for practical applications the timing displacement should be eliminated from the computation of pulse broadening. If the speed of the timing displacement is so slow that the clock recovery can follow the change, then the timing displacement has no influence on system performance and should be eliminated. Accordingly, the pulse width should be defined by

$$T_{w1}{}^{2} = \frac{\int (t - T_c)^2(|u|^2 + |v|^2)\mathrm{d}t}{\int (|u|^2 + |v|^2)\mathrm{d}t}. \tag{10}$$

If the PMD fluctuates more rapidly than the clock recovery can track, it is necessary to include the timing displacement to simulate the transmission system properly:

$$T_{w2}{}^{2} = \frac{\int t^2(|u|^2 + |v|^2)\mathrm{d}t}{\int (|u|^2 + |v|^2)\mathrm{d}t} = T_{w1}{}^{2} + T_c{}^{2}. \tag{11}$$

We compute the mean values, variances, and probability-density functions (PDFs) of these quantities for a general initial pulse u_0 polarized along the u axis with rms width T_0.

4. STOKES VECTOR

In absence of GVD, the Fourier components $\hat{\mathbf{U}} := (\hat{u},\ \hat{v})^T$ of the field,

$$\hat{u}(\omega) = \int u(t)\exp(i\omega t)\mathrm{d}t,$$

$$\hat{v}(\omega) = \int v(t)\exp(i\omega t)\mathrm{d}t,$$

obey a system of ordinary differential equations:

$$\hat{\mathbf{U}}_z = i\omega R(z)\hat{\mathbf{U}}, \tag{12}$$

where the z-dependent matrix R is the random combination [Eq. (3)] of the Pauli matrices. There are simple and exact analytical identities between the amount of broadening and the Fourier components[7,17] as well as among the polarization degree, the time displacement, and the Fourier components. These formulas are in fact nothing more than the standard Parseval formula applied to well-chosen quantities. A convenient representation of the polarization evolutions may be made in terms of Stokes vector $\hat{\mathbf{s}}$ associated with the Fourier components of the field:

$$\hat{s}_1(\omega) = (|\hat{u}|^2 - |\hat{v}|^2)(\omega)/\hat{E}_0(\omega),$$

$$\hat{s}_2(\omega) = 2\,\mathrm{Re}(\hat{u}^*\hat{v})(\omega)/\hat{E}_0(\omega),$$

$$\hat{s}_3(\omega) = 2\,\mathrm{Im}(\hat{u}^*\hat{v})(\omega)/\hat{E}_0(\omega), \tag{13}$$

whose modulus $(\hat{s}_1{}^2 + \hat{s}_2{}^2 + \hat{s}_3{}^2)^{1/2} = (|\hat{u}|^2 + |\hat{v}|^2)/\hat{E}_0(\omega)$ is 1 as we define $\hat{E}_0(\omega)$ as the spectral intensity:

$$\hat{E}_0(\omega) = |\hat{u}|^2(\omega) + |\hat{v}|^2(\omega) = \left|\int u_0(t)\exp(i\omega t)\mathrm{d}t\right|^2, \tag{14}$$

which is a preserved quantity. In terms of the Stokes parameters the dynamics induced by PMD is simple:

$$\hat{\mathbf{s}}_z = 2\sigma\omega\dot{\mathbf{W}}(z) \times \hat{\mathbf{s}}, \tag{15}$$

where $\sigma\dot{\mathbf{W}}(z)$ is the column vector $(m_3,\ m_1,\ m_2)^T(z)$. Thus m_j appear as elementary infinitesimal generators of random rotations of the Stokes vector over the Poincaré sphere. Mathematically speaking, Eq. (15) should be understood as

$$\mathrm{d}\hat{s}_1(\omega) = 2\sigma\omega(\hat{s}_2 \circ \mathrm{d}W^3 - \hat{s}_3 \circ \mathrm{d}W^2),$$

$$\mathrm{d}\hat{s}_2(\omega) = 2\sigma\omega(\hat{s}_3 \circ \mathrm{d}W^1 - \hat{s}_1 \circ \mathrm{d}W^3),$$

$$\mathrm{d}\hat{s}_3(\omega) = 2\sigma\omega(\hat{s}_1 \circ \mathrm{d}W^2 - \hat{s}_2 \circ \mathrm{d}W^1), \tag{16}$$

where $\circ$ stands for the Stratonovich stochastic integral[18] and W^j are three independent Brownian motions. The correlation degree between the Stokes vectors at two nearby frequencies ω_1 and ω_2 is

$$C(\omega_1,\ \omega_2) := \hat{\mathbf{s}}(\omega_1) \cdot \hat{\mathbf{s}}(\omega_2). \tag{17}$$

Applying the standard tools of stochastic analysis (see Appendix A) establishes that C is a diffusion process with an infinitesimal generator:

$$\mathcal{L} = 2\sigma^2\Delta\omega^2\frac{\partial}{\partial C}(1 - C^2)\frac{\partial}{\partial C}, \tag{18}$$

where $\Delta\omega = \omega_1 - \omega_2$. In particular, Eq. (A4) of Appendix A yields that the mean value of C decays exponentially as

$$\langle C(\omega_1,\ \omega_2)\rangle = \exp(-4\Delta\omega^2\sigma^2 z). \tag{19}$$

Equation (19) shows that PMD is a strongly frequency-dependent phenomenon and that the Stokes vectors at two nearby frequencies become statistically independent as z increases. More precisely, we can derive from the expression of the infinitesimal generator the Fokker–Planck equation [Eq. (A5) below] that is satisfied by the PDF $p(z,\ C)$ of the correlation degree $C(\omega_1,\ \omega_2)$ starting from the initial condition $p(0, C) = \delta(C - 1)$. This equation can be solved by means of an expansion over the Legendre polynomials P_n (Ref. 19):

$$p(z,\ C) = \frac{1}{2}\sum_{n=0}^{\infty} P_n(C)\exp[-2n(n + 1)\sigma^2\Delta\omega^2 z]$$

for $-1 \leqslant C \leqslant 1$ [and $p(z,\ C) = 0$ if $|C| > 1$]. The PDF $p(z, \ldots)$ is plotted in Fig. 1 for several values of z. We can thus observe the transition from full correlation $p(0, C) = \delta(C - 1)$ at $z = 0$ to complete uncorrelation $p(\infty,\ C) = 1/2$, $-1 \leqslant C \leqslant 1$, for $\sigma^2\Delta\omega^2 z > 1$.

5. TIME DISPLACEMENT

In terms of the Fourier components the time displacement reads as

$$T_c = \frac{\int \hat{t}(\omega)\hat{E}_0(\omega)\mathrm{d}\omega}{\int \hat{E}_0(\omega)\mathrm{d}\omega}, \tag{20}$$

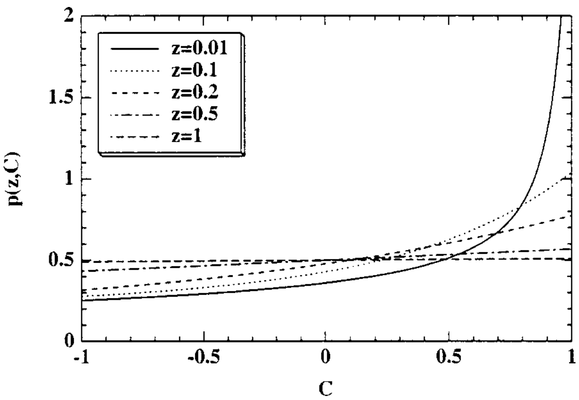

Fig. 1. PDF of correlation degree $C(\omega_1, \omega_2)$ for $\sigma^2\Delta\omega^2 = 1$.

$$\hat{t}(\omega) := \frac{\mathrm{Im}[\hat{u}^*(\omega)\hat{u}'(\omega)] + \mathrm{Im}[\hat{v}^*(\omega)\hat{v}'(\omega)]}{|\hat{u}|^2(\omega) + |\hat{v}|^2(\omega)}, \quad (21)$$

where the prime stands for the derivative with respect to ω. Differentiating $\hat{t}$ with respect to z, we get that the process $\hat{t}(\omega, z)$ as a function of z obeys a simple equation in terms of the Stokes parameters

$$\mathrm{d}\hat{t} = \sigma(\hat{s}_1 \circ \mathrm{d}W^1 + \hat{s}_2 \circ \mathrm{d}W^2 + \hat{s}_3 \circ \mathrm{d}W^3). \quad (22)$$

Note that the right-hand side of Eq. (22) does not depend on $\hat{u}'$ and $\hat{v}'$, which simplifies the forthcoming algebra. Considering the process $(\hat{\mathbf{s}}, \hat{t})$ and applying the methodology presented in Appendix A, we get that $\hat{t}$ is a diffusion process with an infinitesimal generator:

$$\mathcal{L} = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial\hat{t}^2}. \quad (23)$$

Thus $\hat{t}$ obeys the distribution of Brownian motion with a diffusion coefficient σ. Accordingly, the PDF of $\hat{t}(\omega, z)$ is Gaussian:

$$p(\hat{t}) = \frac{1}{\sqrt{2\pi}\sigma\sqrt{z}}\exp\left(-\frac{\hat{t}^2}{2\sigma^2 z}\right).$$

For the following results the correlation function of $\hat{t}$ at two nearby frequencies is necessary. It was found that the process $(C, \hat{t}_1, \hat{t}_2)$, where $\hat{t}_1 = \hat{t}(\omega_1)$, $\hat{t}_2 = \hat{t}(\omega_2)$, and $C = C(\omega_1, \omega_2)$, is a diffusion process with an infinitesimal generator:

$$\mathcal{L} = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial\hat{t}_1{}^2} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial\hat{t}_2{}^2} + \sigma^2 C\frac{\partial^2}{\partial\hat{t}_1\partial\hat{t}_2} + 2\sigma^2\Delta\omega^2\frac{\partial}{\partial C}(1 - C^2)\frac{\partial}{\partial C}. \quad (24)$$

We can thus deduce that $\mathrm{d}\langle\hat{t}_1\hat{t}_2\rangle/\mathrm{d}z = \sigma^2\langle C\rangle$, which yields

$$\langle\hat{t}(\omega_1)\hat{t}(\omega_2)\rangle = \frac{1 - \exp[-4(\omega_1 - \omega_2)^2\sigma^2 z]}{4(\omega_1 - \omega_2)^2}. \quad (25)$$

This result allows us to state that the time displacement has mean zero and variance:

$$\langle T_c{}^2\rangle = \frac{\iint \hat{E}_0(\omega_1)\hat{E}_0(\omega_2)\langle\hat{t}(\omega_1)\hat{t}(\omega_2)\rangle\mathrm{d}\omega_1\mathrm{d}\omega_2}{\left[\int \hat{E}_0(\omega)\mathrm{d}\omega\right]^2}. \quad (26)$$

If the input pulse has a Gaussian shape with rms width T_0, then the variance of the time displacement can be written explicitly:

$$\langle T_c{}^2\rangle = \frac{T_0{}^2}{2}\left[\left(1 + \frac{4\sigma^2 z}{T_0{}^2}\right)^{1/2} - 1\right]. \quad (27)$$

Exact expression (26) depends on the input shape, but the long-term growth rate of the variance is independent of the pulse shape. We can state with great generality that the mean-square time displacement grows linearly for $z \ll T_0{}^2/\sigma^2$, whereas the long-range growth is slower ($\sqrt{z}$):

$$\langle T_c{}^2\rangle \overset{\sigma^2 z \ll T_0{}^2}{\simeq} \sigma^2 z - 4(c_2 - c_4{}^2)\frac{\sigma^4 z^2}{T_0{}^2}$$
$$\overset{\sigma^2 z \gg T_0{}^2}{\simeq} \sqrt{\pi}c_1\sqrt{\sigma^2 z},$$

where

$$c_1 = \frac{\int \hat{E}_0(\omega)^2\mathrm{d}\omega}{\left[\int \hat{E}_0(\omega)\mathrm{d}\omega\right]^2} = \frac{1}{2\pi}\frac{\int |u_0\star u_0|^2(t)\mathrm{d}t}{\left[\int |u_0|^2(t)\mathrm{d}t\right]^2},$$

$$c_2 = \frac{\int \omega^2\hat{E}_0(\omega)\mathrm{d}\omega}{\int \hat{E}_0(\omega)\mathrm{d}\omega} = \frac{\int |u_{0_t}|^2\mathrm{d}t}{\int |u_0|^2\mathrm{d}t},$$

$$c_4 = \frac{\int \omega\hat{E}_0(\omega)\mathrm{d}\omega}{\int \hat{E}_0(\omega)\mathrm{d}\omega} = \frac{i\int u_{0_t}u_0{}^*\mathrm{d}t}{\int |u_0|^2\mathrm{d}t}, \quad (28)$$

and the star stands for convolution. An interesting issue concerns the statistical distribution of T_c. We can show on the one hand that the statistical distribution of T_c is Gaussian when $\sigma^2 z \ll T_0{}^2$ because T_c is then equal to $\sigma W^1(z)$ up to terms of order $\sigma^2 z/T_0{}^2$. On the other hand, when $\sigma^2 z \gg T_0{}^2$ the random variable T_c also has Gaussian statistics because Eqs. (20) and (25) show that it is the sum of a large number of uncorrelated components. The central-limit theorem can then be invoked to show that T_c has Gaussian statistics with zero mean and variance [Eq. (26)]. Accordingly we may think that T_c has a Gaussian PDF in the general case. First note that $[\hat{t}(\omega)]$ as a function of ω is not a Gaussian process, although $\hat{t}(\omega)$ is a Gaussian random variable for each fixed ω. Thus the usual argument that T_c is Gaussian because it is the sum of Gaussian components cannot be invoked. Furthermore, a thorough study shows that the fourth moment of T_c can be expanded as powers of $\sigma^2 z/T_0{}^2$, and the expansion of $\langle T_c{}^4\rangle$ is different from the corresponding expansion of $3\langle T_c{}^2\rangle^2$. This conclusion establishes that

T_c does not have strictly Gaussian statistics, although numerical simulations show that the distribution is close to Gaussian. This resemblance is not surprising, as the expansion of the high-order moments shows that the moments of T_c obey the standard rules of Gaussian random variables[19] up to order 3 with respect to $\sigma^2 z/T_0{}^2$:

$$\langle T_c{}^{2n}\rangle = (2n+1)!!\langle T_c{}^2\rangle^n\left[1 + O\left(\frac{\sigma^6 z^3}{T_0{}^6}\right)\right],$$

where $(2n+1)!! = (2n+1)(2n-1)\ldots 3$. In cases of practical applications for which $\sigma^2 z \leqslant T_0{}^2$, we may thus think of T_c as a Gaussian random variable to a good approximation. In these conditions the PDF of T_c is

$$p(t) = \frac{1}{\sqrt{2\pi}\langle T_c{}^2\rangle^{1/2}}\exp\left(-\frac{t^2}{2\langle T_c{}^2\rangle}\right),$$

where $\langle T_c{}^2\rangle$ is given by Eq. (26) in the general case and by Eq. (27) in the case of a Gaussian pulse.

6. PULSE WIDTH

In terms of the Fourier components, pulse width T_{w2} reads as

$$T_{w2}{}^2 = \frac{\int \hat{R}(\omega)\hat{E}_0(\omega)\mathrm{d}\omega}{\int \hat{E}_0(\omega)\mathrm{d}\omega}, \tag{29}$$

$$\hat{R}(\omega) := \frac{|\hat{u}'|^2(\omega) + |\hat{v}'|^2(\omega)}{|\hat{u}|^2(\omega) + |\hat{v}|^2(\omega)}. \tag{30}$$

The process $\hat{R}(\omega, z)$ as a function of z obeys a stochastic differential equation that reads as

$$\mathrm{d}\hat{R} = \sigma(\hat{r}_1 \circ \mathrm{d}W^1 + \hat{r}_2 \circ \mathrm{d}W^2 + \hat{r}_3 \circ \mathrm{d}W^3), \tag{31}$$

where the vector $\hat{\mathbf{r}}(\omega)$,

$$\hat{r}_1(\omega) = 2\,\mathrm{Im}(\hat{u}'\hat{u}^* - \hat{v}'\hat{v}^*)/\hat{E}_0(\omega),$$

$$\hat{r}_2(\omega) = 2\,\mathrm{Im}(\hat{u}'\hat{v}^* + \hat{v}'\hat{u}^*)/\hat{E}_0(\omega),$$

$$\hat{r}_3(\omega) = 2\,\mathrm{Re}(\hat{u}'\hat{v}^* - \hat{v}'\hat{u}^*)/\hat{E}_0(\omega), \tag{32}$$

is solution of

$$\hat{\mathbf{r}}_z = 2\sigma\omega\dot{\mathbf{W}}(z) \times \hat{\mathbf{r}} + 2\sigma\dot{\mathbf{W}}(z). \tag{33}$$

Vector $\hat{\mathbf{r}}$ is the so-called PMD vector. Let us denote $\tau(\omega, z) = 4[\hat{R}(\omega, z) - \hat{R}(\omega, 0)]$. Differentiating $|\hat{\mathbf{r}}(\omega, z)|^2 - \tau(\omega, z)$ with respect to z establishes that it is a constant; thus $\tau = |\hat{\mathbf{r}}|^2$ is the so-called square DGD. Furthermore, τ is a diffusion process with an infinitesimal generator:

$$\mathcal{L} = 8\sigma^2\tau\frac{\partial^2}{\partial\tau^2} + 12\sigma^2\frac{\partial}{\partial\tau}, \tag{34}$$

which implies that $\tau(\omega, z)$ obeys a χ^2 distribution with three degrees of freedom, also known as a Maxwell distribution. In other words, the PDF of $\tau(\omega, z)$ is

$$p(\tau) = \frac{\tau^{1/2}}{\sqrt{2\pi}(4\sigma^2 z)^{3/2}}\exp\left(-\frac{\tau}{8\sigma^2 z}\right), \qquad \tau \geqslant 0.$$

Note that the pulse width $T_{w2}(z)$ is always larger than $T_{w2}(0)$, whatever z and whatever the imperfections of the fiber, because $\tau(z, \omega) \geqslant 0$. Besides, the statistical distribution of τ is independent of ω. In particular, its mean grows linearly with length: $\langle\tau\rangle = 12\sigma^2 z$. The correlation function of τ at two nearby frequencies is required for the forthcoming statements. It was found that the process (τ_1, τ_2, C_p), where $\tau_1 = \tau(\omega_1)$, $\tau_2 = \tau(\omega_2)$, and $C_p = \hat{\mathbf{r}}(\omega_1)\cdot\hat{\mathbf{r}}(\omega_2)$, is a diffusion process with an infinitesimal generator:

$$\begin{aligned}\mathcal{L} = {}& 8\sigma^2\tau_1\frac{\partial^2}{\partial\tau_1{}^2} + 12\sigma^2\frac{\partial}{\partial\tau_1} + 8\sigma^2\tau_2\frac{\partial^2}{\partial\tau_2{}^2} + 12\sigma^2\frac{\partial}{\partial\tau_2} \\ & + 16\sigma^2 C_p\frac{\partial^2}{\partial\tau_1\partial\tau_2} + \sigma^2(12 - 4\Delta\omega^2 C_p)\frac{\partial}{\partial C_p} \\ & + 2\sigma^2[\Delta\omega^2(\tau_1\tau_2 - C_p{}^2) + \tau_1 + \tau_2 + 2C_p]\frac{\partial^2}{\partial C_p{}^2} \\ & + 8\sigma^2(C_p + \tau_1)\frac{\partial^2}{\partial C_p\partial\tau_1} + 8\sigma^2(C_p + \tau_2)\frac{\partial^2}{\partial C_p\partial\tau_2},\end{aligned} \tag{35}$$

where $\Delta\omega = \omega_2 - \omega_1$. The complete expression for the generator represents a straightforward generalization of previously known results. For instance, it allows us to recover the expression for the expectation of the degree of correlation between the PMD vectors at two nearby frequencies:

$$\langle C_p\rangle = \frac{3}{\Delta\omega^2}[1 - \exp(-4\sigma^2\Delta\omega^2 z)], \tag{36}$$

which was first derived in Ref. 5. We can also compute the covariance of the square DGD at two nearby frequencies, ω_1 and ω_2:

$$\mathrm{Cov}(\tau_1, \tau_2) = \frac{12}{\Delta\omega^4}[4\Delta\omega^2\sigma^2 z - 1 + \exp(-4\Delta\omega^2\sigma^2 z)], \tag{37}$$

which is consistent with the formula derived in Ref. 6. Further, we can compute the expected value of any combination of C_p, τ_1, and τ_2. For instance, the second moment of the correlation degree of C_p is found to be

$$\begin{aligned}\langle C_p{}^2\rangle = {}& \frac{1}{24\Delta\omega^2}[4 + 24\Delta\omega^2\sigma^2 z + 72\Delta\omega^4\sigma^4 z^2 \\ & - 9\exp(-4\Delta\omega^2\sigma^2 z) + 5\exp(-12\Delta\omega^2\sigma^2 z)].\end{aligned} \tag{38}$$

Once the statistical distribution of τ is known it is easy to compute the mean and the variance of $T_{w2}{}^2$:

$$\langle T_{w2}{}^2\rangle = T_0{}^2 + 3\sigma^2 z, \tag{39}$$

where T_0 is the initial pulse width. Note that the linear growth of the square pulse width is a universal feature that does not depend on the shape of the initial pulse.

This remarkable property is derived from the independence of the expected value of τ with respect to ω. However, the variance of $T_{w2}{}^2$ depends on the shape of the initial pulse:

$$\mathrm{Var}(T_{w2}{}^2) = \frac{\int\int \hat{E}_0(\omega_1)\hat{E}_0(\omega_2)\mathrm{Cov}[\tau(\omega_1), \tau(\omega_2)]\mathrm{d}\omega_1\mathrm{d}\omega_2}{16\left[\int \hat{E}_0(\omega)\mathrm{d}\omega\right]^2}.$$

In the case of a Gaussian pulse

$$\mathrm{Var}(T_{w2}{}^2) = T_0(T_0{}^2 + 4\sigma^2 z)^{3/2} - 6T_0{}^2\sigma^2 z - T_0{}^4. \tag{40}$$

More generally, the following identity is valid whatever the input pulse:

$$\frac{\mathrm{d}\,\mathrm{Var}(T_{w2}{}^2)}{\mathrm{d}z} = 12\sigma^2\langle T_c{}^2\rangle. \tag{41}$$

We can then deduce some properties of the variance:

$$\mathrm{Var}(T_{w2}{}^2) \overset{\sigma^2 z \ll T_0{}^2}{\simeq} 6\sigma^4 z^2$$
$$\overset{\sigma^2 z \gg T_0{}^2}{\simeq} 8\sqrt{\pi}c_1(\sigma^2 z)^{3/2}.$$

Let us now compute the PDF of $T_{w2}{}^2$:

$$\int_{t^2}^{\infty} p(u)\mathrm{d}u = \mathbb{P}(T_{w2}{}^2 \geq t^2).$$

It can be readily estimated in the asymptotic configuration: $\sigma^2 z \ll T_0{}^2$ or $\sigma^2 z \gg T_0{}^2$. If $\sigma^2 z \ll T_0{}^2$, then the processes $\tau(\omega)$ are completely correlated for ω in the spectrum of the pulse, so $T_{w2}{}^2 - T_0{}^2$ is the sum of the squares of three independent Gaussian random variables and its PDF is

$$p(t^2) \overset{\sigma^2 z \ll T_0{}^2}{=} \frac{\sqrt{t^2 - T_0{}^2}}{\sqrt{2\pi}(\sigma^2 z)^{3/2}}\exp\left(-\frac{t^2 - T_0{}^2}{2\sigma^2 z}\right), \qquad t^2 \geq T_0{}^2.$$

If $\sigma^2 z \gg T_0{}^2$, then Eq. (37) shows that the frequency correlation radius of the process τ (or $\hat{R}$) is of the order of $(\sigma^2 z)^{-1/2}$, which is much smaller than the bandwidth $T_0{}^{-1}$ of $\hat{E}_0$. Thus $T_{w2}{}^2$ is the sum of a large number of uncorrelated components. The central-limit theorem can then be invoked to establish that the statistics of $T_{w2}{}^2$ are Gaussian. Because the mean value and the variance of $T_{w2}{}^2$ are known, the PDF of $T_{w2}{}^2$ is, for $t^2 \geq T_0{}^2$,

$$p(t^2) \overset{\sigma^2 z \ll T_0{}^2}{=} \frac{1}{\sqrt{2\pi\,\mathrm{Var}(T_{w2}{}^2)}}\exp\left[-\frac{(t^2 - \langle T_{w2}{}^2\rangle)^2}{2\,\mathrm{Var}(T_{w2}{}^2)}\right].$$

Note that the decorrelation rate between $\tau(\omega_1)$ and $\tau(\omega_2)$ is very slow [as $(\omega_1 - \omega_2)^{-2}$], so the second regime can be observed only when $\sigma^2 z$ is much larger than $T_0{}^2$. Besides, this argument shows that the first expression of the PDF is still valid even for $\sigma^2 z \sim T_0{}^2$, if we take care to take into account the exact expression of the variance of $T_{w2}{}^2$,

$$p(t^2) \overset{\sigma^2 z \ll T_0{}^2}{\simeq} \frac{\sqrt{t^2 - T_2{}^2}}{\sqrt{2\pi}\gamma_2{}^{3/2}}\exp\left(-\frac{t^2 - T_2{}^2}{2\gamma_2}\right), \qquad t^2 \geq T_2{}^2,$$

where

$$\gamma_2 = [\mathrm{Var}(T_{w2}{}^2)/6]^{1/2},$$
$$T_2{}^2 = \langle T_{w2}{}^2\rangle - [3\,\mathrm{Var}(T_{w2}{}^2)/2]^{1/2}.$$

If we consider T_{w1}, the results read as follows: The mean value of $T_{w1}{}^2$ is simply the difference between the mean values of $T_{w2}{}^2$ and $T_c{}^2$. The variance of $T_{w1}{}^2$ is

$$\mathrm{Var}(T_{w1}{}^2) = \mathrm{Var}(T_{w2}{}^2) + \mathrm{Var}(T_c{}^2) - 2\,\mathrm{Cov}(T_{w2}{}^2, T_c{}^2).$$

For a Gaussian pulse we get

$$\langle T_{w1}{}^2\rangle = T_0{}^2 + 3\sigma^2 z - \frac{T_0{}^2}{2}\left[\left(1 + \frac{4\sigma^2 z}{T_0{}^2}\right)^{1/2} - 1\right]. \tag{42}$$

Note that this expression was also derived in Ref. 5. The variance of $T_{w1}{}^2$ is

$$\mathrm{Var}(T_{w1}{}^2) = T_0{}^4\left(1 + \frac{4\sigma^2 z}{T_0{}^2}\right)^{1/2}\left[\left(1 + \frac{4\sigma^2 z}{T_0{}^2}\right)^{1/2} - 1\right]^2. \tag{43}$$

For a general pulse shape, the mean value of $T_{w1}{}^2$ grows linearly:

$$\langle T_{w1}{}^2\rangle \overset{\sigma^2 z \ll T_0{}^2}{\simeq} T_0{}^2 + 2\sigma^2 z \tag{44}$$
$$\overset{\sigma^2 z \gg T_0{}^2}{\simeq} 3\sigma^2 z, \tag{45}$$

whereas the variance satisfies

$$\mathrm{Var}(T_{w1}{}^2) \overset{\sigma^2 z \ll T_0{}^2}{\simeq} 4\sigma^4 z^2$$
$$\overset{\sigma^2 z \gg T_0{}^2}{\simeq} 8\sqrt{\pi}c_1(\sigma^2 z)^{3/2}.$$

Let us now estimate the PDF of $T_{w1}{}^2$. If $\sigma^2 z \ll T_0{}^2$, then we know that $T_{w1}{}^2 = T_0{}^2 + \sigma^2 W^2(z)^2 + \sigma^2 \times W^3(z)^2 + O(\sigma^4 z^2/T_0{}^2)$, so the PDF obeys a χ^2 distribution with two degrees of freedom (i.e., a Rayleigh distribution):

$$p(t^2) \overset{\sigma^2 z \ll T_0{}^2}{=} \frac{1}{2\sigma^2 z}\exp\left(-\frac{t^2 - T_0{}^2}{2\sigma^2 z}\right), \qquad t^2 > T_0{}^2. \tag{46}$$

If $\sigma^2 z \gg T_0{}^2$, then $T_{w2}{}^2$ is the prevailing term in the expression of $T_{w1}{}^2$, so their PDFs are the same. In the general configuration, $T_{w2}{}^2$ obeys a χ^2 distribution with three degrees of freedom (i.e., a Maxwellian distribution), whereas $T_c{}^2$ is the square of a Gaussian random variable, so we can guess that the PDF of $T_{w1}{}^2$ corresponds to a χ^2 distribution with a number d_1 of degrees of freedom that is 2–3. Because we know the mean and the variance of $T_{w1}{}^2$ we can fix the free parameters:

$$d_1 = (\langle T_{w1}{}^2\rangle - T_0{}^2)/(\sigma^2 z),$$

such that, for $t^2 > T_1{}^2$,

$$p(t^2) = \frac{(t^2 - T_1{}^2)^{d_1/2-1}}{2^{d_1/2}\Gamma(d_1/2)\gamma_1{}^{d_1/2}}\exp\left(-\frac{t^2 - T_1{}^2}{2\gamma_1}\right), \quad (47)$$

where $\Gamma(s) = \int_0^\infty t^{s-1}\exp(-t)\mathrm{d}t$ and

$$\gamma_1 = [\mathrm{Var}(T_{w1}{}^2)/(2d_1)]^{1/2},$$

$$T_1{}^2 = \langle T_{w1}{}^2\rangle - [d_1\,\mathrm{Var}(T_{w1}{}^2)/2]^{1/2}.$$

7. ROTATION OF POLARIZATION

The rotation of the pulse polarization can be characterized by the ratio P_r of the energy in the input polarization state to the total energy. In terms of the first Stokes parameter the parameter P_r reads as

$$P_r = \frac{1}{2} + \frac{1}{2}\frac{\int \hat{s}_1(\omega)\hat{E}_0(\omega)\mathrm{d}\omega}{\int \hat{E}_0(\omega)\mathrm{d}\omega}. \quad (48)$$

The computation of the mean and the covariance of $\hat{s}_1$ is straightforward:

$$\langle \hat{s}_1(\omega)\rangle = \exp(-4\omega^2\sigma^2 z),$$

$$\mathrm{Cov}[\hat{s}_1(\omega_1)\hat{s}_1(\omega_2)]$$
$$= 1/3\exp[-4(\omega_1 - \omega_2)^2\sigma^2 z] + 2/3\exp[-4(\omega_1{}^2 + \omega_2{}^2 + \omega_1\omega_2)\sigma^2 z] - \exp[-4(\omega_1{}^2 + \omega_2{}^2)\sigma^2 z].$$

We can then state that the mean of P_r is

$$\langle P_r\rangle = \frac{1}{2} + \frac{1}{2}\frac{\int \exp(-4\omega^2\sigma^2 z)\hat{E}_0(\omega)\mathrm{d}\omega}{\int \hat{E}_0(\omega)\mathrm{d}\omega}, \quad (49)$$

which for a Gaussian pulse is equal to

$$\langle P_r\rangle = \frac{1}{2} + \frac{1}{2}\left(1 + \frac{2\sigma^2 z}{T_0{}^2}\right)^{-1/2}. \quad (50)$$

For a general pulse we have

$$\langle P_r\rangle \overset{\sigma^2 z \ll T_0{}^2}{\simeq} 1 - 2\sigma^2 z c_2$$
$$\overset{\sigma^2 z \gg T_0{}^2}{\simeq} \frac{1}{2} + \frac{c_3\sqrt{\pi}}{4\sqrt{\sigma^2 z}},$$

where

$$c_3 = \frac{\hat{E}_0(0)}{\int \hat{E}_0(\omega)\mathrm{d}\omega} = \frac{1}{2\pi}\frac{\left|\int u_0\mathrm{d}t\right|^2}{\int |u_0|^2\mathrm{d}t}.$$

The fluctuations about the mean value can be estimated by computation of the variance of P_r:

$$\mathrm{Var}(P_r)$$
$$= \frac{\iint \mathrm{Cov}[\hat{s}_1(\omega_1),\hat{s}_1(\omega_2)]\hat{E}_0(\omega_1)\hat{E}_0(\omega_2)\mathrm{d}\omega_1\mathrm{d}\omega_2}{4\left[\int \hat{E}_0(\omega)\mathrm{d}\omega\right]^2}.$$

For a Gaussian pulse,

$$\mathrm{Var}(P_r) = \frac{1}{12}\left(1 + \frac{4\sigma^2 z}{T_0{}^2}\right)^{-1/2} - \frac{1}{4}\left(1 + \frac{2\sigma^2 z}{T_0{}^2}\right)^{-1} + \frac{1}{6}\left(1 + \frac{3\sigma^2 z}{T_0{}^2}\right)^{-1/2}\left(1 + \frac{\sigma^2 z}{T_0{}^2}\right)^{-1/2}. \quad (51)$$

For a general pulse we have

$$\mathrm{Var}(P_r) \overset{\sigma^2 z \ll T_0{}^2}{\simeq} 16\sigma^4 z^2 c_2{}^2$$
$$\overset{\sigma^2 z \gg T_0{}^2}{\simeq} \frac{\sqrt{\pi}c_1}{6\sqrt{\sigma^2 z}}.$$

8. DEGREE OF POLARIZATION

The degree of polarization P_d can be expressed in terms of the frequency correlation degree of the Stokes vectors as

$$P_d = \frac{\left(\iint C(\omega_1,\,\omega_2)\hat{E}_0(\omega_1)\hat{E}_0(\omega_2)\mathrm{d}\omega_1\mathrm{d}\omega_2\right)^{1/2}}{\int \hat{E}_0(\omega)\mathrm{d}\omega}. \quad (52)$$

It is then easy to compute the mean of $P_d{}^2$. We then get that the degree of polarization and the time displacement satisfy the equation

$$\frac{\mathrm{d}\langle T_c{}^2\rangle}{\mathrm{d}z} = \sigma^2\langle P_d{}^2\rangle. \quad (53)$$

For a Gaussian pulse we have

$$\langle P_d{}^2\rangle = \left(1 + \frac{4\sigma^2 z}{T_0{}^2}\right)^{-1/2}. \quad (54)$$

For a general pulse the degree of polarization first decays linearly:

$$\langle P_d{}^2\rangle \overset{\sigma^2 z \ll T_0{}^2}{\simeq} 1 - 8\sigma^2 z(c_2 - c_4{}^2),$$

whereas it slowly decays to 0 for long propagation distances:

$$\langle P_d{}^2\rangle \overset{\sigma^2 z \gg T_0{}^2}{\simeq} \frac{\sqrt{\pi}c_1}{2\sigma\sqrt{z}},$$

where c_j are defined by Eqs. (28). More exactly, if $\sigma^2 z \ll T_0{}^2$, then $C \simeq 1 - 2\sigma^2(\omega_2 - \omega_1)^2[W^2(z)^2 + W^3(z)^2]$, so the PDF of the random variable $1 - P_d$ corresponds to a Rayleigh distribution:

$$p(x) = \frac{1}{4\sigma^2(c_2 - c_4{}^2)z}\exp\left[-\frac{x}{4\sigma^2(c_2 - c_4{}^2)z}\right].$$

If $\sigma^2 z \gg T_0{}^2$, then $P_d{}^2$ is the sum of the squares of three quasi-independent Gaussian random variables $\int \hat{s}_j(\omega)\hat{E}_0(\omega)\mathrm{d}\omega$, so the PDF of P_d is

$$p(x) = \frac{\sqrt{2}}{\sqrt{\pi}\alpha^{3/2}} x^2 \exp\left(-\frac{x^2}{2\alpha}\right),$$

where $\alpha(z) = (\sqrt{\pi}c_1)/(6\sigma\sqrt{z})$.

9. INFLUENCE OF GROUP-VELOCITY DISPERSION

If we take into account both the GVD and the PMD, then the field $\mathbf{U} = (u, v)^T$ satisfies Eq. (2). In terms of the Fourier components $\hat{\mathbf{U}} = (\hat{u}, \hat{v})^T$ this partial differential equation reads as a collection of ordinary differential equations:

$$\hat{\mathbf{U}}_z = i\omega R(z)\hat{\mathbf{U}} + i\frac{\beta''}{2}\omega^2\hat{\mathbf{U}}. \quad (55)$$

Setting $\check{\mathbf{U}} = \hat{\mathbf{U}}\exp(-i\beta''\omega^2 z/2)$ yields a system of differential equations for $\check{\mathbf{U}}$ that is the same as in the absence of GVD. The relevant quantities $\hat{E}_0(\omega)$, $\hat{R}(\omega)$, and $\hat{t}(\omega)$ can be expressed as

$$\hat{E}_0(\omega) = \check{E}_0(\omega),$$

$$\hat{R}(\omega) = \check{R}(\omega) + \beta''^2\omega^2 z^2 + 2\beta''\omega z\check{t}(\omega),$$

$$\hat{t}(\omega) = \check{t}(\omega) + \beta''\omega z,$$

where $\check{E}_0$, $\check{R}$, and $\check{t}$ are defined in terms of $\check{u}$ and $\check{v}$ similarly as $\hat{E}_0$, $\hat{R}$, and $\hat{t}$. We know the statistical distributions of $\check{E}_0$, $\check{R}$, and $\check{t}$, from the arguments presented in the previous sections, so it is easy to get the statistical distributions of $\hat{E}_0$, $\hat{R}$, and $\hat{t}$.

Let us first address the timing displacement. Only the mean value of the time displacement is modified by the presence of GVD:

$$\langle T_c\rangle = \beta'' c_4 z, \qquad \mathrm{Var}(T_c) = \mathrm{Var}(T_c)|_{\beta''=0}.$$

For a Gaussian pulse we have $c_4 = 0$, which shows that GVD has no influence on the timing displacement. Let us now consider pulse broadening. Both the mean value and the variance of the pulse width are enhanced by GVD:

$$\langle T_{w2}{}^2\rangle = T_0{}^2 + 3\sigma^2 z + c_2\beta''^2 z^2,$$

$$\mathrm{Var}(T_{w2}{}^2) = \mathrm{Var}(T_{w2}{}^2)|_{\beta''=0} + 4\beta''^2 z^2 \times \frac{\int\int \omega_1\omega_2\langle\check{t}(\omega_1)\check{t}(\omega_2)\rangle\check{E}_0(\omega_1)\check{E}_0(\omega_2)\mathrm{d}\omega_1\mathrm{d}\omega_2}{\left[\int \check{E}_0(\omega)\mathrm{d}\omega\right]^2},$$

$$\langle T_{w1}{}^2\rangle = T_0{}^2 + 3\sigma^2 z + (c_2 + c_4{}^2)\beta''^2 z^2 + \mathrm{Var}(T_c)|_{\beta''=0},$$

$$\mathrm{Var}(T_{w1}{}^2) = \mathrm{Var}(T_{w2}{}^2) - 2\,\mathrm{Var}(T_c)^2.$$

For the Gaussian pulse we have

$$\langle T_{w2}{}^2\rangle = T_0{}^2 + 3\sigma^2 z + \frac{\beta''^2 z^2}{4T_0{}^2}$$

$$\mathrm{Var}(T_{w2}{}^2) = T_0{}^4\left[\left(1 + \frac{4\sigma^2 z}{T_0{}^2}\right)^{3/2} - 6\frac{\sigma^2 z}{T_0{}^2} - 1\right] + \frac{\beta''^2 z^2}{4}\frac{\{[1 + (4\sigma^2 z/T_0{}^2)]^{1/2} - 1\}^2}{[1 + (4\sigma^2 z/T_0{}^2)]^{1/2}}$$

$$\langle T_{w1}{}^2\rangle = T_0{}^2 + 3\sigma^2 z + \frac{\beta''^2 z^2}{4T_0{}^2} - \frac{T_0{}^2}{2}\left[\left(1 + \frac{4\sigma^2 z}{T_0{}^2}\right)^{1/2} - 1\right]$$

$$\mathrm{Var}(T_{w1}{}^2) = T_0{}^4\left(1 + \frac{4\sigma^2 z}{T_0{}^2}\right)^{1/2}\left[\left(1 + \frac{4\sigma^2 z}{T_0{}^2}\right)^{1/2} - 1\right]^2 \times \left[1 + \frac{\beta''^2 z^2}{4T_0{}^4[1 + (4\sigma^2 z/T_0{}^2)]}\right].$$

The degree of polarization is not affected by GVD.

10. NUMERICAL SIMULATIONS

Here we solve the propagation equation numerically in absence of GVD. We consider a Gaussian pulse or a sech

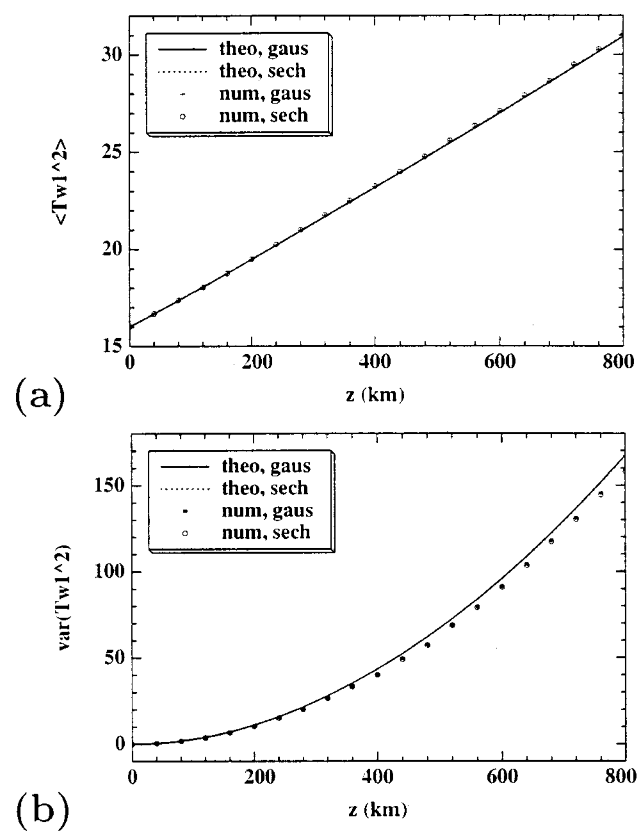

Fig. 2. Square rms widths of pulses with Gaussian (gaus) or sech shapes. Curves, theoretical (theo) values; crosses and circles, results averaged over 10^4 numerical (num) simulations. (a) Mean values; (b) variances.

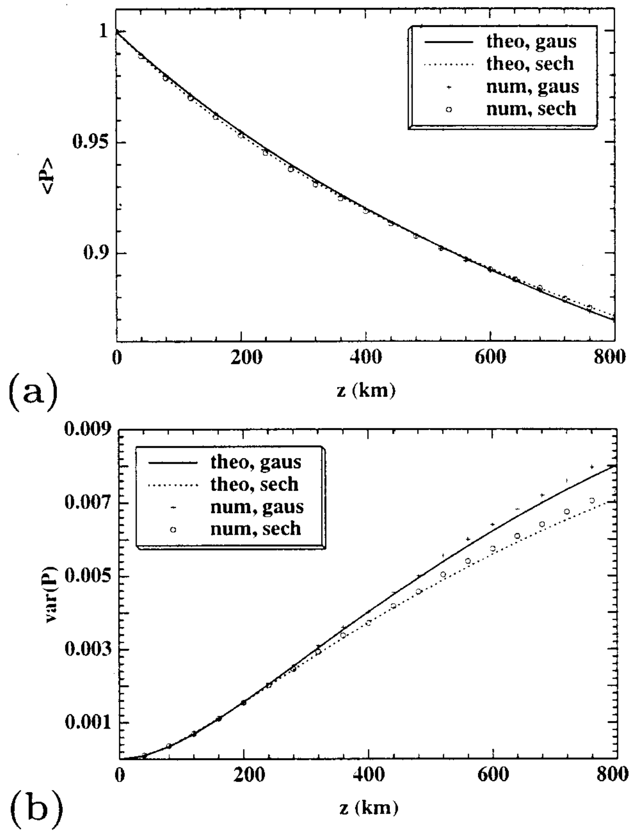

Fig. 3. Rotations of the polarization of pulses with Gaussian or sech shapes. Curves, theoretical values; crosses and circles, results averaged over 10^4 numerical simulations. (a) Mean values; (b) variances. Abbreviations as for Fig. 2.

pulse as an initial condition with an initial rms width $T_0 = 4$ ps. We assume that the magnitude of linear birefringence is constant. To simulate the random fluctuations of birefringence we incorporate, at fictitious junctions between adjacent fiber pieces with length Δz, random axial rotation and the addition of a random phase difference between the field components. We take $\Delta\beta' = 2$ ps/km and $\Delta z = 25$ m, so $\sigma^2 = 8.3 \times 10^{-3}$ ps^2/km. Note that this configuration corresponds to one with the usual parameter $D_p = 0.29$ ps/$\sqrt{\text{km}}$. The number of samples used for evaluating average values and variances is 10^4. Comparisons of numerical results and theoretical formulas show excellent agreement (Figs. 2 and 3).

Another series of runs was carried out in which the local birefringence strength and angle varied with distance. We present in Figs. 4 and 5 the results that correspond to two configurations for which $\sigma^2 z/T_0{}^2 = 0.12$ ($\ll 1$) and $\sigma^2 z/T_0{}^2 = 2.2$, respectively. The agreement with the theoretical formulas is still excellent. In particular, the numerical histograms of the pulse widths are found to be close to the theoretical PDF.

11. CONCLUSION

In this paper we have studied the behavior of short pulses in fibers with randomly varying birefringence. We have derived closed-form expressions for the statistical values of relevant quantities. The mean values, variances, and PDFs of time displacement, time rms width, and degree of polarization have been computed. The results give further insight in to the PMD dynamic as well as practical help for engineers in that not only the mean values averaged over an ensemble of fibers are given. As was remarked in Ref. 6, the statistical properties of PMD in long fibers are uniquely determined by the mean DGD. It could be interesting also to take into account polarization-dependent losses. This could certainly be done, as the statistical distribution of the DGD was carried out in Ref. 20.

A. STOCHASTIC CALCULUS

We review some basic results of the probability theory.[18] Let us consider a smooth vector field $M:\mathbb{R}^n \to \mathbb{R}^{n\times d}$ and a d-dimensional Brownian motion W (which means that the derivatives $\dot{W}^j$, $j = 1,\ldots, d$, are independent Gaussian white noises). Define the $\mathbb{R}^n$-valued random process X solution of the stochastic differential equation:

$$\frac{\mathrm{d}X_i(z)}{\mathrm{d}z} = \sum_{j=1}^{d} M_{ij}[X(z)]\dot{W}^j(z), \qquad i = 1,\ldots, n, \tag{A1}$$

which is equivalent to the Stratonovich integral form $X(z) = X(0) + \int_0^z M[X(s)] \circ \mathrm{d}W(s)$. $X(z)$ is a diffusion process with infinitesimal generator $\mathcal{L}$:

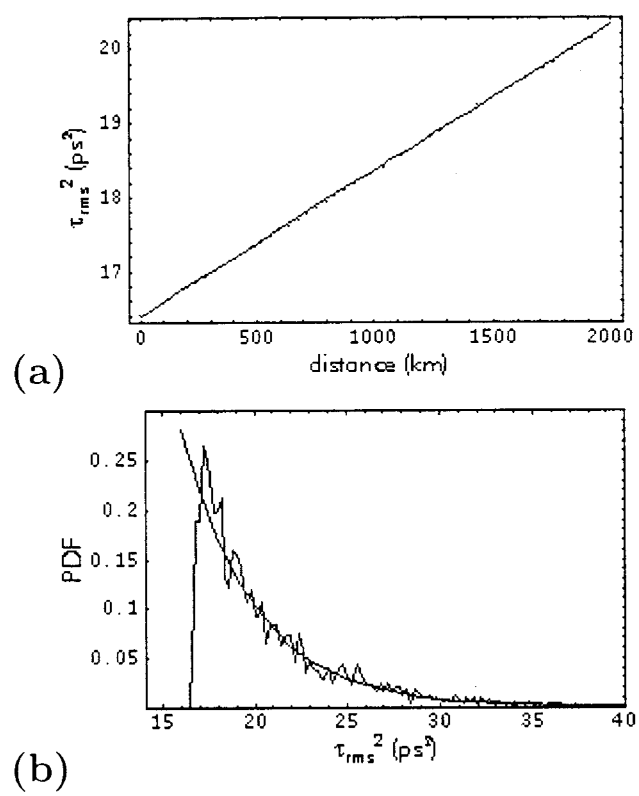

Fig. 4. (a) Mean-square width and (b) PDF of the pulse width of pulses with Gaussian shapes. The numerical values were computed from 3000 runs. The PDF is plotted at $z = 2000$ km. $T_0{}^2 = 16.4$ ps^2 and $D_p = 0.1$ ps/$\sqrt{\text{km}}$. The PDF is plotted at $z = 2000$ km, which corresponds to $\sigma^2 z/T_0{}^2 = 0.12$. The theoretical PDF is Eq. (46).

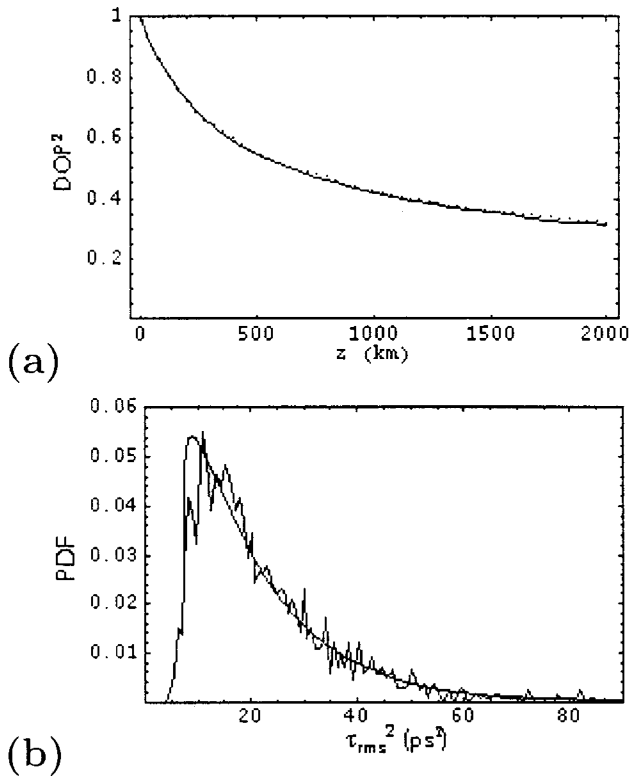

Fig. 5. (a) Degree of polarization and (b) PDF of the pulse width of pulses with Gaussian shapes. The numerical values were computed from 3000 runs. ${T_0}^2 = 3.6\ \mathrm{ps}^2$ and $D_p = 0.2\ \mathrm{ps}/\sqrt{\mathrm{km}}$. The PDF is plotted at $z = 2000$ km, which corresponds to $\sigma^2 z/{T_0}^2 = 2.2$. The theoretical PDF is Eq. (47).

$$\mathcal{L} = \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \tag{A2}$$

$$a_{ij}(x) = \sum_{k=1}^{d} M_{ik}(x) M_{jk}(x), \tag{A3}$$

$$b_i(x) = \sum_{j=1}^{n} \sum_{k=1}^{d} \frac{\partial M_{ik}(x)}{\partial x_j} M_{jk}(x). \tag{A4}$$

This implies that any expectation of a smooth function $f(X)$ of X satisfies the forward Kolmogorov equation,

$$\frac{\partial \langle f(X) \rangle}{\partial z} = \langle \mathcal{L} f(X) \rangle, \tag{A5}$$

and that the PDF $p(z, x) = \langle \delta[X(z) - x] \rangle$ of the process $X(z)$ satisfies the Fokker–Planck equation

$$\frac{\partial p}{\partial z} = \mathcal{L}^* p, \tag{A6}$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$:

$$\mathcal{L}^* pk = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} [b_i(x) p] + \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} [a_{ij}(x) p]. \tag{A7}$$

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